General announcements

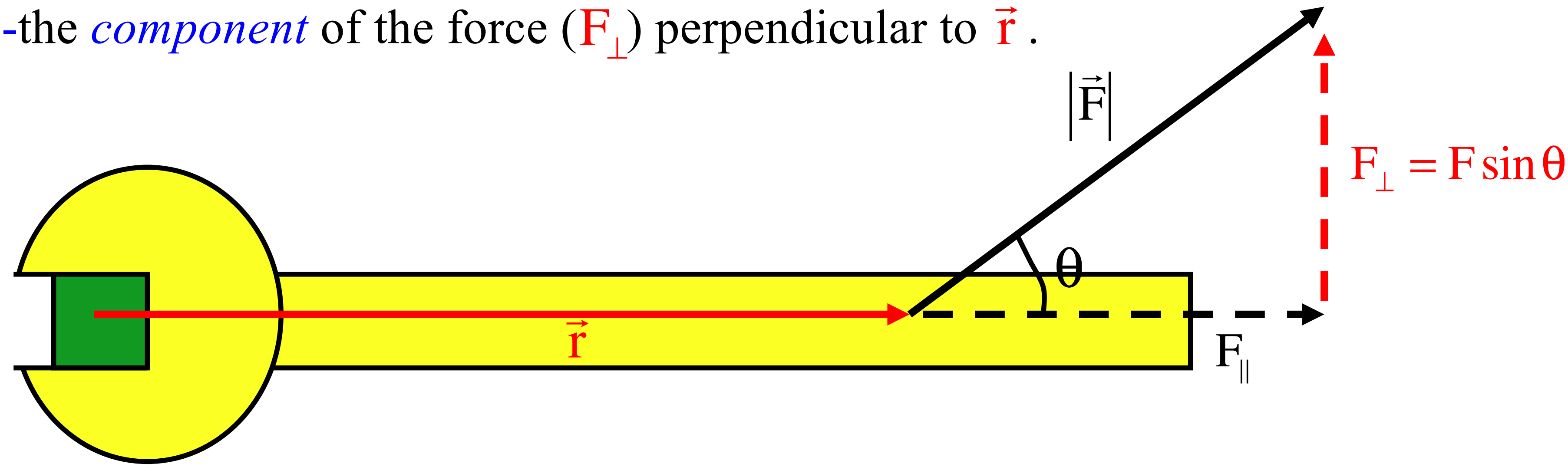
Torque

Consider the wrench shown below. Notice that the amount of rotational umph (this is a technical term) provided by the wrench on the bolt depends upon:

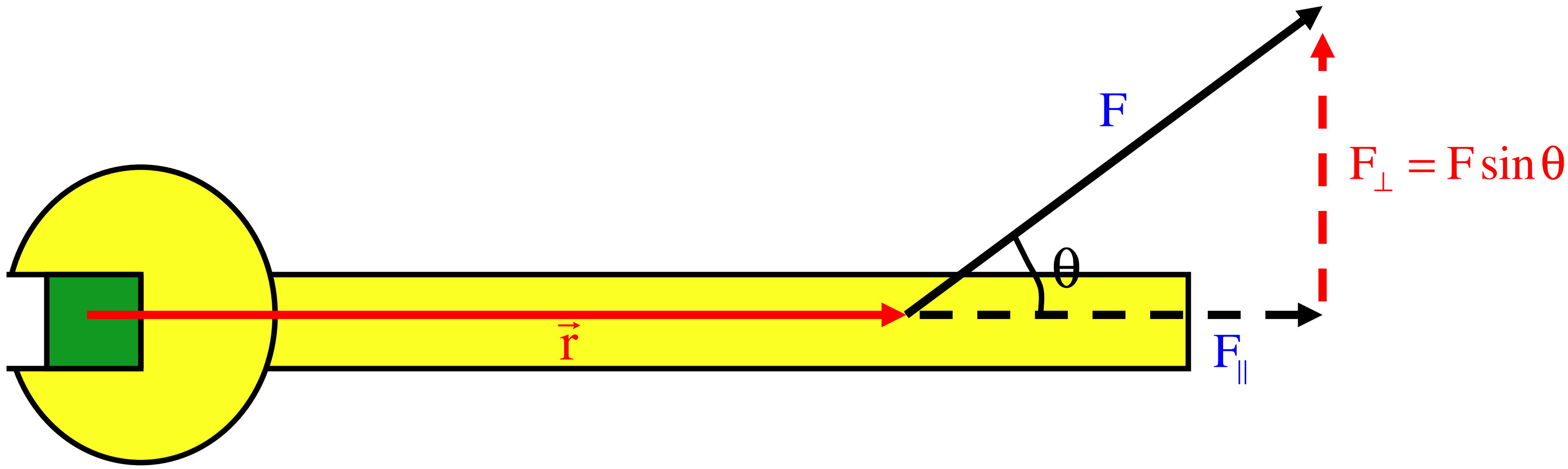
--how far out the force is applied ($\vec{r}$), and

--how big the force is ($|\vec{F}|$), and

--the *component* of the force ($F_\perp$) perpendicular to $\vec{r}$.



The product of $F_\perp$ and $|\vec{r}|$ generates the rotational counterpart to force, a vector called *torque*. When a net *torque* is applied to a stationary object that is free to rotate, the object will *angularly accelerate*.



So formally defined, the MAGNITUDE of the torque $|\vec{\tau}|$ generated by the force on the wrench is mathematically equal to: $|\vec{\tau}| = |\vec{r}|\ F_\perp$

As can be seen in the graphic, the perpendicular component of the force is equal to:

$$F_\perp = F\sin\theta$$

where θ is defined as the angle between *the line of the force* and *the line of the position vector* $\vec{r}$. (This definition is going to be important later.)

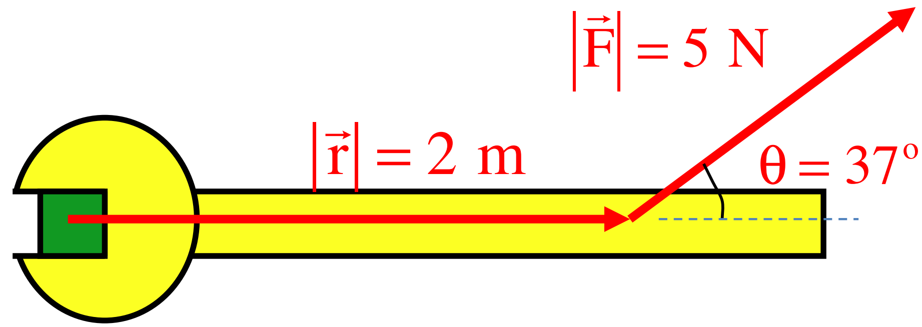
This means the torque can also be written as:

$$\begin{aligned}|\vec{\tau}| &= |\vec{r}|\ F_\perp \\ &= |\vec{r}||\vec{F}|\sin\theta\end{aligned}$$

In fact, there are three ways to calculate a torque using polar information:

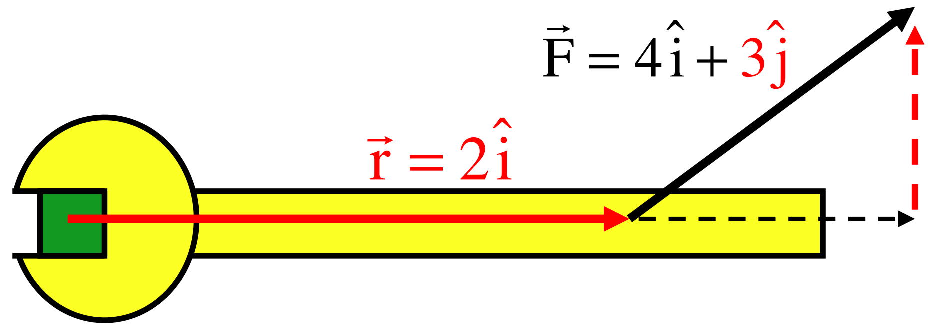
Definition approach:

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$= (2\text{ m})(5\text{ N})\sin 37^\circ$$
$$= 6\text{ N}\bullet\text{m}$$



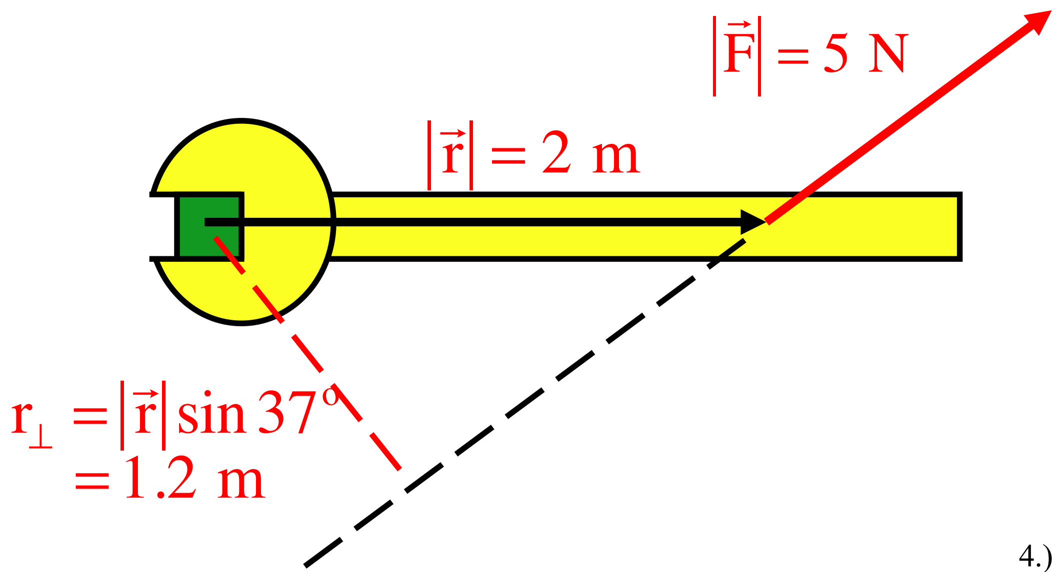
F-perpendicular approach:

$$\left|\vec{\tau}\right| = F_{\perp}\left|\vec{r}\right|$$
$$= (3\text{ N})(2\text{ m})$$
$$= 6\text{ N}\bullet\text{m}$$



r-perpendicular approach:

$$\left|\vec{\tau}\right| = r_{\perp}\left|\vec{F}\right|$$
$$= (1.2\text{ m})(5\text{ N})$$
$$= 6\text{ N}\bullet\text{m}$$



Why is the *r-perpendicular approach* so powerful (and mostly preferred)?

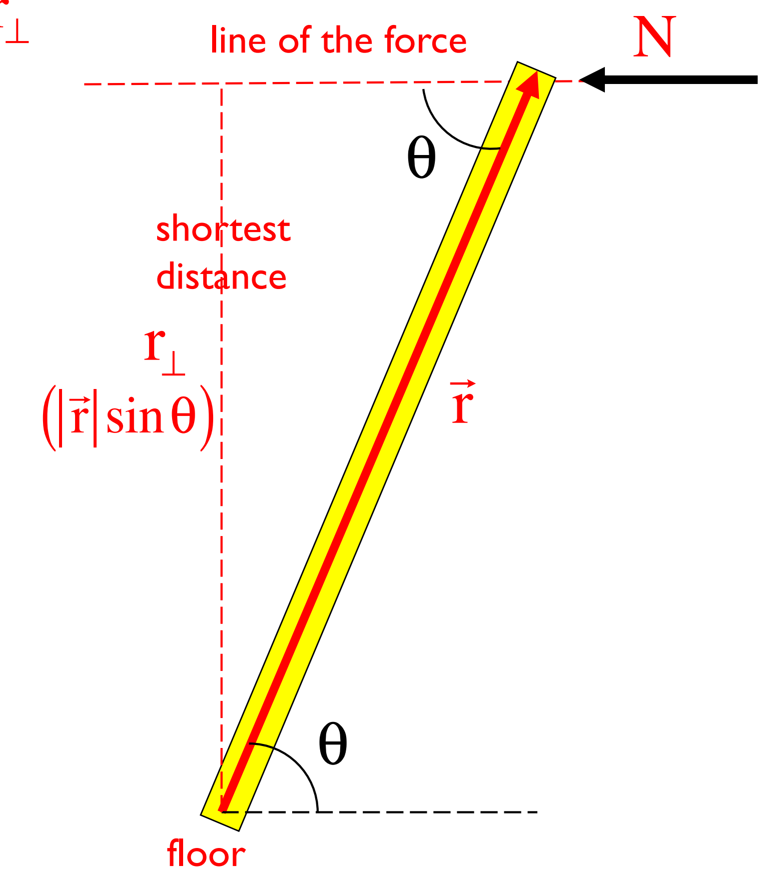
Can you find the *shortest distance between a point and a line*? If so, you can find $r_\perp$ for any situation. Called the moment arm, that is what $r_\perp$ is, the shortest distance between the *point about which you are taking the torque* and *the line of the force* (and you don't really need to define $\vec{r}$ to do it!) With it, $|\vec{\tau}| = r_\perp |\vec{F}|$.

Example 6: A ladder sits against a wall. Using the $r_\perp$ approach, determine the torque generated by the wall's normal force *N* about the ladder's contact with the floor.

1. Identify the *line of N*.

2. Identify shortest distance between *floor contact* and *line of N*, then express in terms of whatever parameters are handy (this will often be in terms of $|\vec{r}|$).

3. Determine the torque with $|\vec{\tau}| = r_\perp |\vec{F}|$.



Side point: *Although we started* by looking at a wrench, there are all sorts of instances in physics when we want to product of the magnitude of one vector and the perpendicular component of the second vector.

Because it pops up so often, this process is called *a cross product*. For two vectors $\vec{C}$ and $\vec{D}$, the magnitude of $\vec{D}x\vec{C}$ is defined such that:

$$\left|\vec{D}x\vec{C}\right| = \left|\vec{D}\right|\left|\vec{C}\right|\sin\phi,$$

where ϕ is the angle between *the line of* $\vec{C}$ and *the line of* $\vec{D}$.

The direction of a cross product will be *perpendicular to the plane* determined by $\vec{C}$ and $\vec{D}$, and can be determined using the *right hand rule*.

Note that if $\vec{r}$ and $\vec{F}$ are in the x-y plane, the direction of the *cross product* will be in the + or – $\hat{k}$-direction.

This is all fine and swell if you are dealing with polar notation, but what about vectors in *unit vector notation*? Specifically, if:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then $\vec{B}x\vec{A} = \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)x\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$

$$= \left[\left(B_x\hat{i}\right)x\left(A_x\hat{i}\right)\right] + \left[\left(B_y\hat{j}\right)x\left(A_x\hat{i}\right)\right] + \ldots$$

But $\left(B_x\hat{i}\right)x\left(A_x\hat{i}\right) = B_xA_x \sin 0^0 = 0$, so all the like-terms go to zero, and $\left(B_y\hat{j}\right)x\left(A_x\hat{i}\right) = B_yA_x \sin 90^0 = B_yA_x$ in the -k direction, so we end up with 6 non-zero parts.

What's interesting is that those six parts fall out with the evaluation of the matrix:

$$\vec{B}x\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

Question is, how do you evaluate a matrix like this?

The operation is fairly simple (something you do over and over again).

$$\vec{B}x\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ B_x & B_y \\ A_x & A_y \end{matrix}$$

$$= \hat{i}\left[\left(B_y\right)\left(A_z\right) - \left(B_z\right)\left(A_y\right)\right]$$

Blank out the column and row in which exists the unit vector $\hat{i}$.

Evaluate the two-by-two matrix that is to the immediate right, and multiply it by $\hat{i}$.

Adding in the $\hat{j}$ term looks like:

$$\vec{B} x \vec{A} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ B_x & B_y & B_z & B_x & B_y \\ A_x & A_y & A_z & A_x & A_y \end{array} \right.$$

$$= \hat{i}\left[(B_y)(A_z) - (B_z)(A_y)\right] + \hat{j}\left[(B_z)(A_x) - (B_x)(A_z)\right]$$

Finishing off with the $\hat{k}$ term gives us:

$$\vec{B} x \vec{A} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ B_x & B_y & B_z & B_x & B_y \\ A_x & A_y & A_z & A_x & A_y \end{array} \right.$$

$$= \hat{i}\left[(B_y)(A_z) - (B_z)(A_y)\right] + \hat{j}\left[(B_z)(A_x) - (B_x)(A_z)\right] + \hat{k}\left[(B_x)(A_y) - (B_y)(A_x)\right]$$

DON'T MEMORIZE the end result. *Know the approach*!

Example 8: Determine $\vec{D} x \vec{C}$ if:

$$\vec{C} = (-1)\hat{i} + (2)\hat{j}$$

$$\vec{D} = (4)\hat{i} + (5)\hat{j} + (6)\hat{k}$$

$$\vec{D} x \vec{C} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 4 & 5 & 6 & 4 & 5 \\ -1 & 2 & 0 & -1 & 2 \end{array} \right.$$

Example 8: (con't) Determine $\vec{D}x\vec{C}$ if:

$$\vec{C} = (-1)\hat{i} + (2)\hat{j}$$

$$\vec{D} = (4)\hat{i} + (5)\hat{j} + (6)\hat{k}$$

Solution:

$$\vec{D}x\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix}$$

$$\vec{D}x\vec{C} = \hat{i}\left[(5)(0)-(6)(2)\right] + \hat{j}\left[(6)(-1)-(4)(0)\right] + \hat{k}\left[(4)(2)-(5)(-1)\right]$$

$$= \quad -12\hat{i} \quad -6\hat{j} \quad +13\hat{k}$$

Torque's Analogue (a review)

We've seen rotational analogues for x, v, a (**θ**, **ω**, **α**)

What would be the translational analogue for torque?

– Hint: what does a torque produce?

Thinking back to the wrench example, a torque produces a rotation – if the wrench starts at rest, we obviously see that the torque changes the motion of the wrench. Hang on…

A (net) torque produces an angular acceleration. What does that sound like?

<u>This tells us that torque is the rotational analogue to force!</u>

<u>If the net torque on an object is 0, the object's angular motion will not change—it will NOT angularly accelerate.</u>

Cross products (Ms. Dunham's)

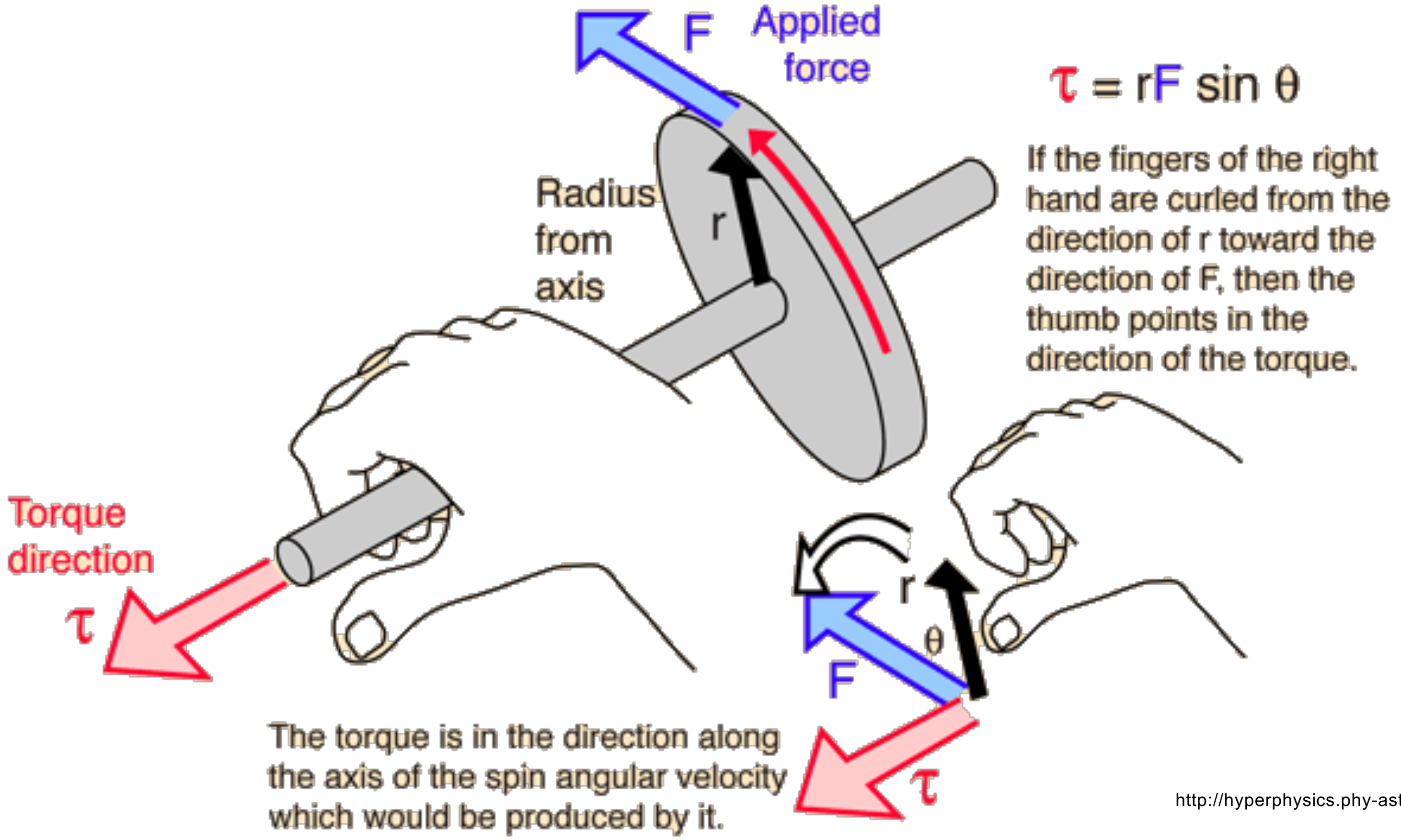
- A cross product is useful in physics for many situations, even though right now we're focusing on torque. It's important to know the general rules for calculating cross products in polar notation.
- Remember that a cross product can be simplified to:

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|sin\theta$$

  - Like a dot product, the angle in that equation represents the **angle between vectors A and B**. If you are given two vectors in polar notation, you need to find the angle between them, not just plug in one or the other of the given angles immediately!
- This gives us magnitude. What about direction?

Cross product directions

- Unlike a dot product (which produces a scalar), a cross product produces a vector (which has direction)
- The **direction** of a cross product can be found by the right hand rule. <u>The order of the vectors matters!</u>



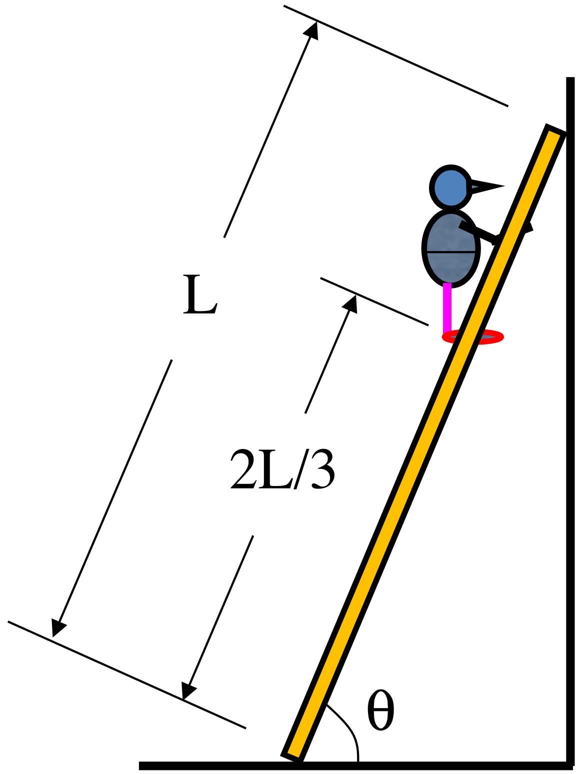
http://hyperphysics.phy-astr.gsu.edu/hbase/tord.html

NOTE: In a conceptual sense, if a force on a body produces a torque about some point, it means *the force is motivating the body to angularly accelerate about that point*.

Put a little differently, if you do a torque calculation for a force acting on a body and get zero, it means the force will NOT motivate the body to change its rotational motion about the point in question.

BOTTOM LINE in a computational sense, if the line of a force passes through the point about which you are taking a torque, that force will provide NO TORQUE about that point (which is to say, it will not motivate the body to angularly accelerate about that point).

A ladder of length L and mass m_L sits at an angle θ. A man of mass m_m stands a distance $2L/3$ meters up its length. If the wall is frictionless, what are the forces acting at the wall and at the floor.



According to the f.b.d., we have three unknowns. Although this is NOT what you will probably do first on your test, the most obvious source of equations is to use the translational version of N.S.L. in the x and y-directions. Noting that the acceleration is zero everywhere, we can write:

$$\underline{\sum F_x :}$$

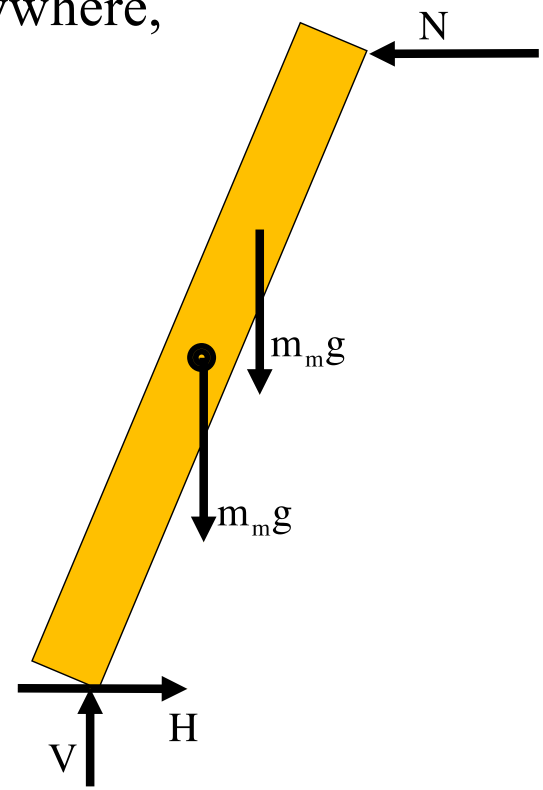
$$H - N = m\cancel{a}_x^{\,=0}$$

$$\Rightarrow \quad H = N$$

$$\underline{\sum F_y :}$$

$$V - m_L g - m_m g = m\cancel{a}_y^{\,=0}$$

$$\Rightarrow \quad V = m_L g + m_m g$$



Three unknowns and two equations--we need another equation. That will come from summing the torques about a convenient point and put that equal to zero (as the angular acceleration about any point on the ladder will be zero).

Torque on the ladder

For educational purposes, we are going to do torque calculations relative to two points, one about the *center of mass* and the other about the *contact point* with the floor.

We also have several ways to do the torque calculations:

1. the definition approach (r-cross-F);
2. the "r-perpendicular" approach (i.e., the moment-arm approach); and
3. the "F-perpendicular" approach.

Using all three, we will begin with the calculation relative to the *center of mass* of the ladder.

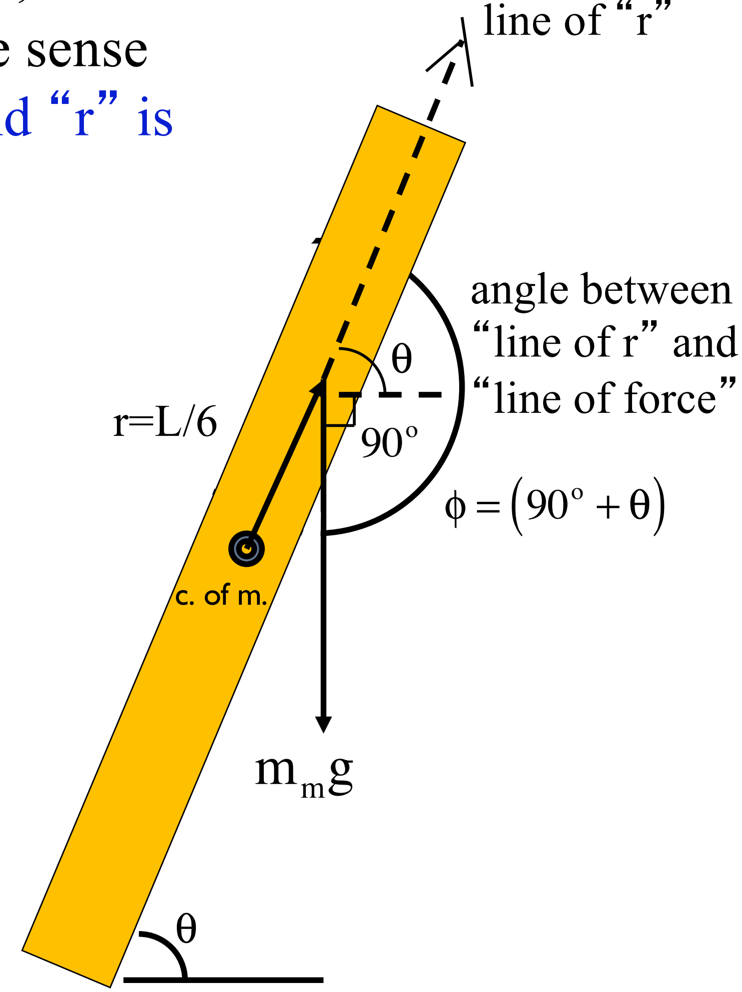
We will do this in pieces so you can follow the drift, then I'll put it all together and write it out in the way you would on a test (assuming you'd used this particular path on the test, which is not likely as it is more complicated than need be . . . but, again, educational, so read it!).

a.) *The LADDER' S WEIGHT* acts through the ladder's *center of mass*, so the torque due to that force about the *center of mass* will be zero. (This makes sense as that force will provide no rotational impetus for rotation about the *center of mass*.)

b.) *The torque* generated by the MAN' S WEIGHT, relative to the *center of mass*, is difficult only in the sense that determine the angle between the line of "F" and "r" is not obvious. The diagram should help. Using the "definition" approach:

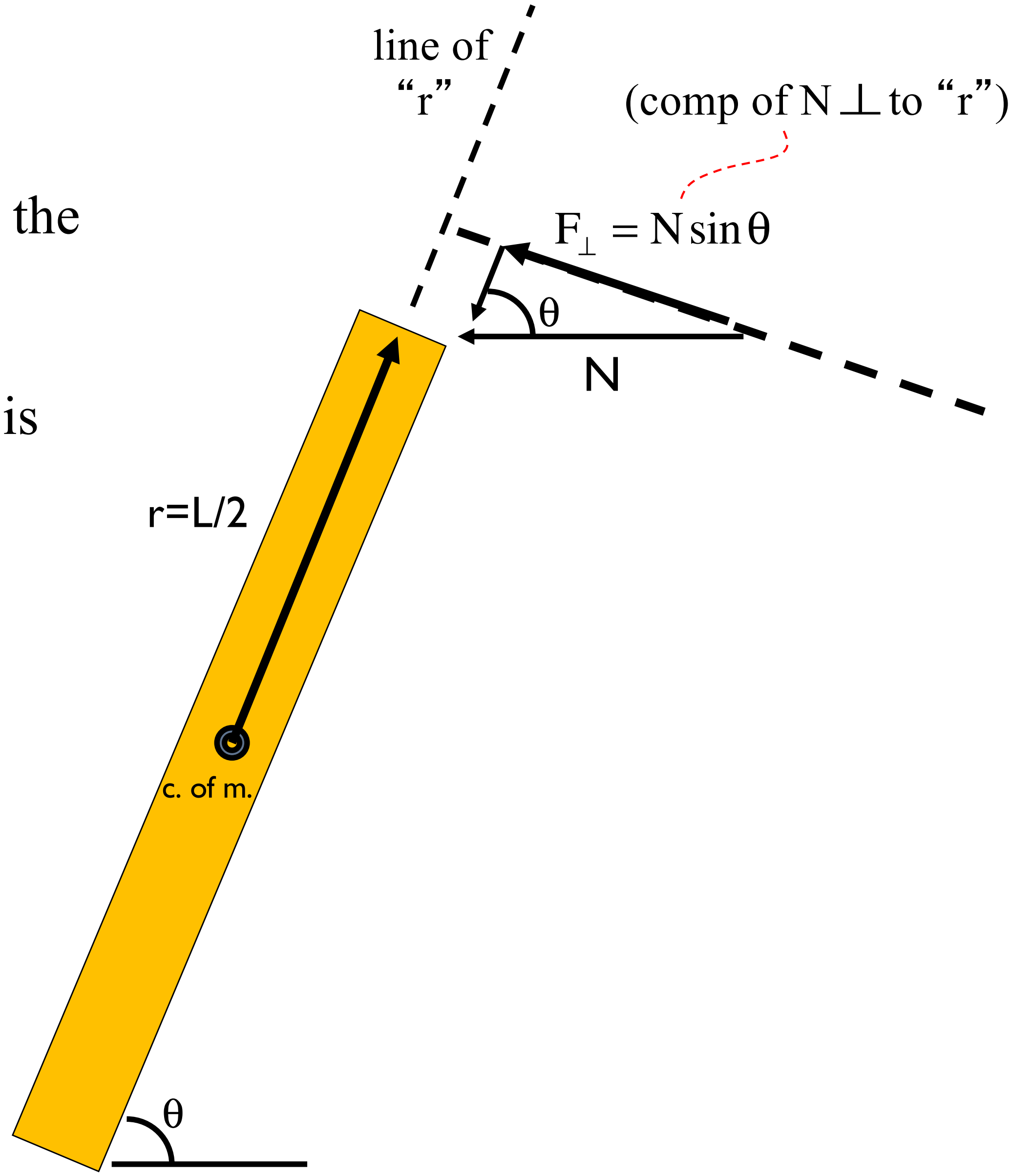
$$\Gamma_{m_m g} = \pm \ |F| \ |r| \ \sin\phi$$

$$= -(m_m g)\left(\frac{L}{6}\right)\sin(90^\circ + \theta)$$



Note: the torque is negative because the force involved is trying to motivate the ladder to rotate in a CLOCKWISE direction, relative to the center of mass.

c.) *The torque* generated by the NORMAL at the wall, relative to the center of mass, is the next stop. Using the "F-perpendicular" approach (i.e., the product of the magnitude of the "r" vector and the component of "F" that is perpendicular to "r"), we get:

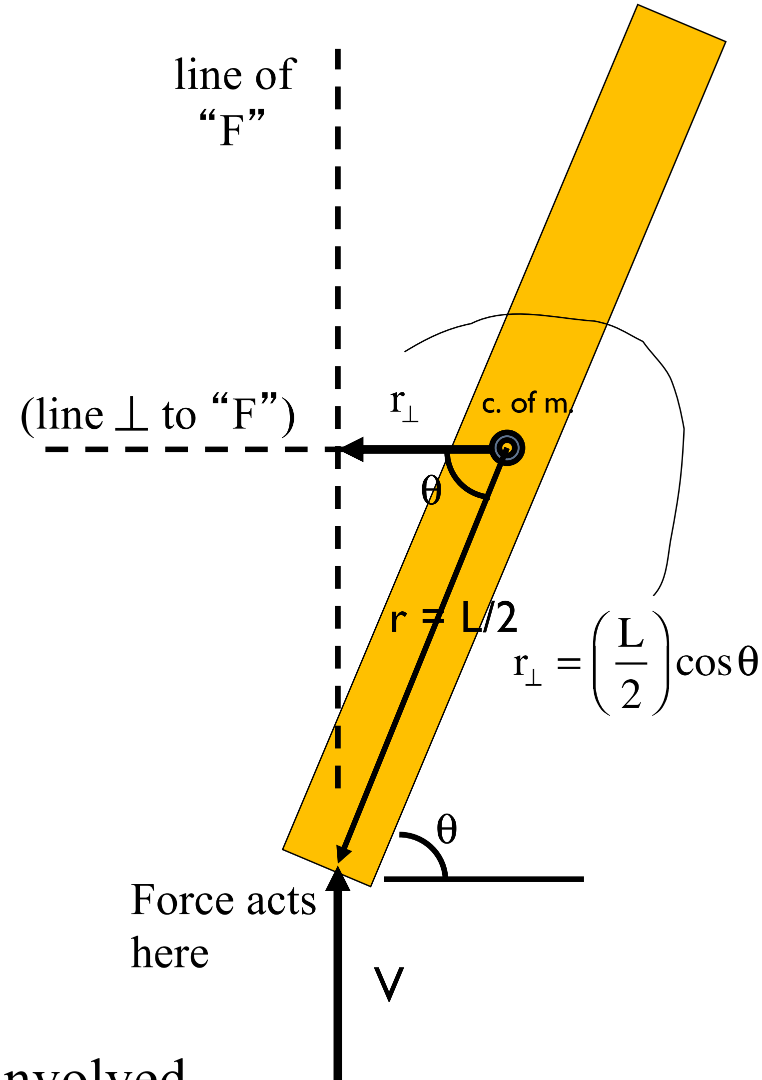


$$\Gamma_N = \pm \; |F_\perp| \quad |r|$$

$$= +(N\sin\theta)\left(\frac{L}{2}\right)$$

Note: the torque is positive because the force involved is trying to motivate the ladder to rotate in a COUNTER-CLOCKWISE direction, relative to the *center of mass*.

c.) *Next is the torque* generated by the normal force "V" at the floor, relative to the center of mass. Using the "r-perpendicular" approach (i.e., the product of the magnitude of the "F" vector and the component of "r" that is perpendicular to "F"--this is called "the moment arm," by the way), we get:



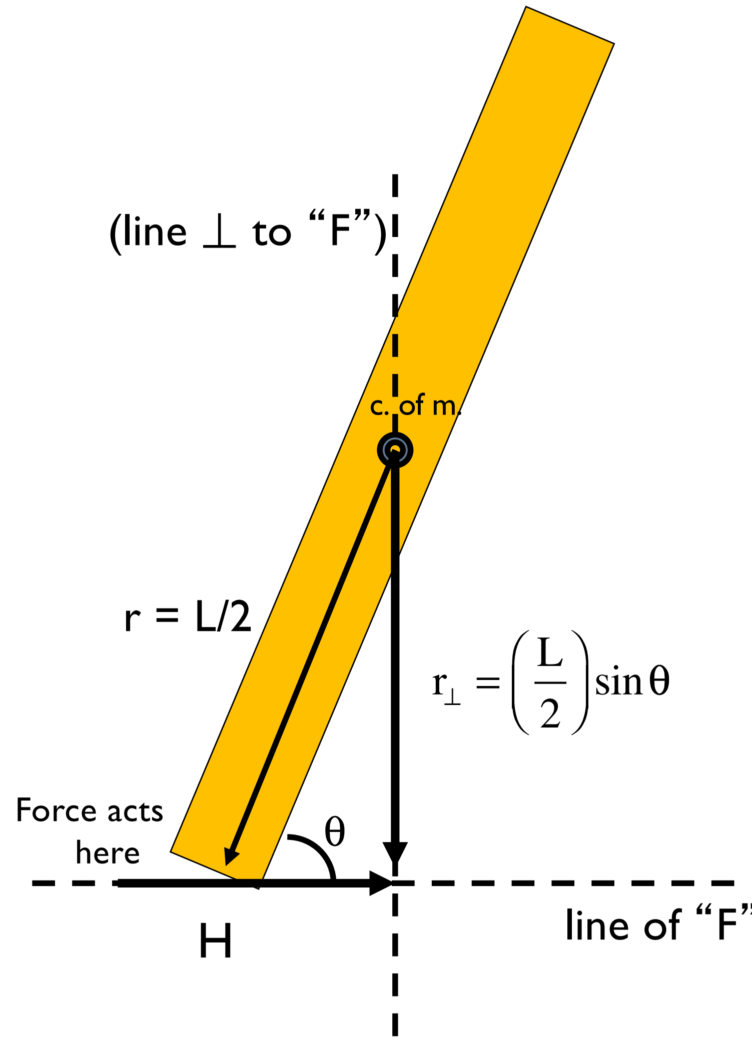
$$\Gamma_N = \pm\ |F|\quad |r_\perp|$$

$$= -(V)\left(\frac{L}{2}\cos\theta\right)$$

Note: the torque is negative because the force involved is trying to motivate the ladder to rotate in a CLOCKWISE direction, relative to the *center of mass*.

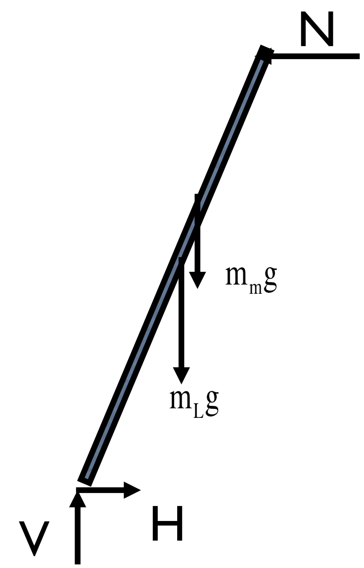
c.) *Next is the torque* generated by the frictional force "H" at the floor, relative to the center of mass. Using the "r-perpendicular" approach again (i.e., the product of the magnitude of the "F" vector and the component of "r" that is perpendicular to "F"--this is called "the moment arm," by the way), we get:

$$\Gamma_N = \pm|F| \quad |r_\perp|$$
$$= +(H)\left(\frac{L}{2}\sin\theta\right)$$



Note: the torque is positive because the force involved is trying to motivate the ladder to rotate in a COUNTERCLOCKWISE direction, relative to the *center of mass*.

c.) *Therefore, the sum* of the torques (with the unknowns in red for easy identification) looks like:



$\underline{\sum \Gamma_{cm}}$:

$$\left|\Gamma_{m_L g}\right| + \left|\Gamma_{m_m g}\right| + \left|\Gamma_N\right| + \left|\Gamma_H\right| + \left|\Gamma_V\right| = \cancelto{0}{I\alpha}$$

$$0 + \left[-(m_m g)\left(\frac{L}{6}\right)\sin\left(90^\circ + \theta\right)\right] + \left[(N\sin\theta)\left(\frac{L}{2}\right)\right] + \left[H\left(\frac{L}{2}\sin\theta\right)\right] + \left[-V\left(\frac{L}{2}\cos\theta\right)\right] = 0$$

$$|\Gamma_{m_L g}| + |\Gamma_{m_m g}| + |\Gamma_N| + |\Gamma_H| + |\Gamma_V| = I\cancel{\alpha}^{=0}$$

$$0 + \left[-(m_m g)\left(\frac{L}{6}\right)\sin\left(90^{o}+\theta\right)\right] + \left[(N\sin\theta)\left(\frac{L}{2}\right)\right] + \left[H\left(\frac{L}{2}\sin\theta\right)\right] + \left[-V\left(\frac{L}{2}\cos\theta\right)\right] = 0$$

From before, "$N = H$" and "$V = m_L g + m_m g$." As $\sin(90^{o}+\theta) = \cos\theta$, we can write:

$$-(m_m g)\left(\frac{L}{6}\right)\sin\left(90^{o}+\theta\right) + (N\sin\theta)\left(\frac{L}{2}\right) + H\left(\frac{L}{2}\sin\theta\right) - V\left(\frac{L}{2}\cos\theta\right) = 0$$

$$\Rightarrow \quad -(m_m g)\left(\frac{\cancel{L}}{6}\right)\cos\theta + (N\sin\theta)\left(\frac{\cancel{L}}{2}\right) + N\left(\frac{\cancel{L}}{2}\sin\theta\right) - (m_L g + m_m g)\left(\frac{\cancel{L}}{2}\cos\theta\right) = 0$$

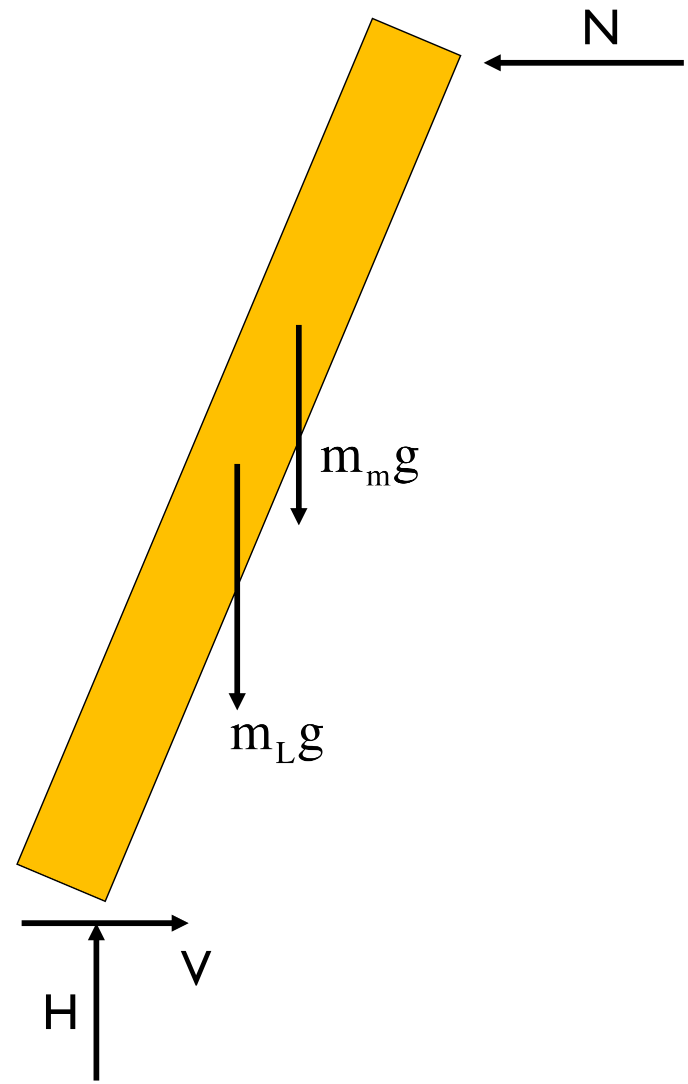
$$\Rightarrow \quad -\left(\frac{m_m g}{6}\right)\cos\theta - \left(\frac{m_L g + m_m g}{2}\cos\theta\right) = -\left(\frac{N}{2}\right)\sin\theta - \left(\frac{N}{2}\right)\sin\theta$$

$$\Rightarrow \quad \left(\frac{m_m g}{6} + \frac{m_L g + m_m g}{2}\right)\cos\theta = N\sin\theta$$

$$\Rightarrow \quad \left(\frac{m_m g}{6} + \frac{m_L g + m_m g}{2}\right)\frac{\cos\theta}{\sin\theta} = \left(\frac{m_m g}{6} + \frac{m_L g}{2} + \frac{m_m g}{2}\right)\frac{\cos\theta}{\sin\theta} = N$$

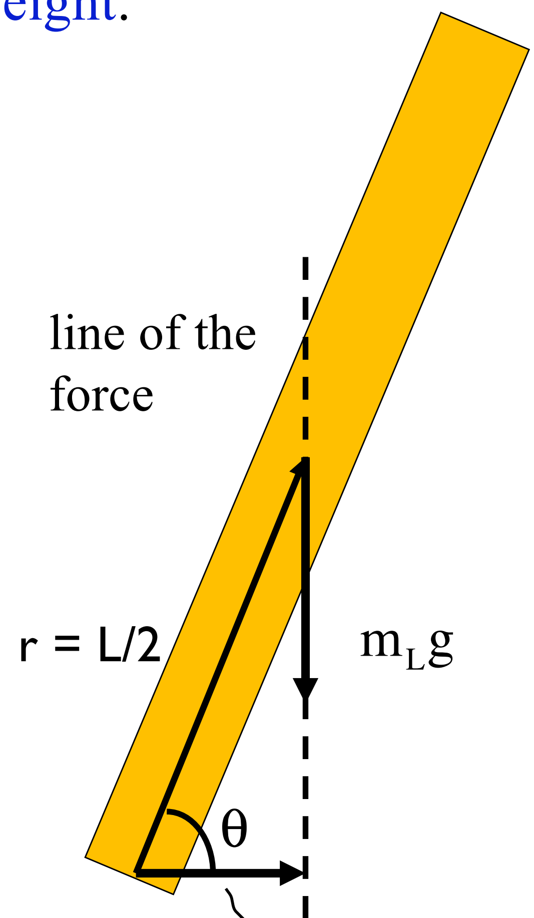
$$\Rightarrow \quad N = \left(\frac{2m_m g}{3} + \frac{m_L g}{2}\right)\cot\theta \qquad (= H)$$

There is an easier way to do all of this. We still have to sum the forces, but if we sum the torques about the *floor* instead of the *center of mass* the torques due to "H" and "V" will be zero and we can ignore them. That leaves torques for only two weights and "N." Using the "moment arm" approach (i.e., the r-perpendicular approach), each calculations will be presented below, then the whole things summarized as you would present the material on a test.



To get the torque due to the ladder's weight:

$$\Gamma_{m_L g} = -\ |F|\ \ \ |r_\perp|$$
$$= -(m_L g)\left(\frac{L}{2}\cos\theta\right)$$



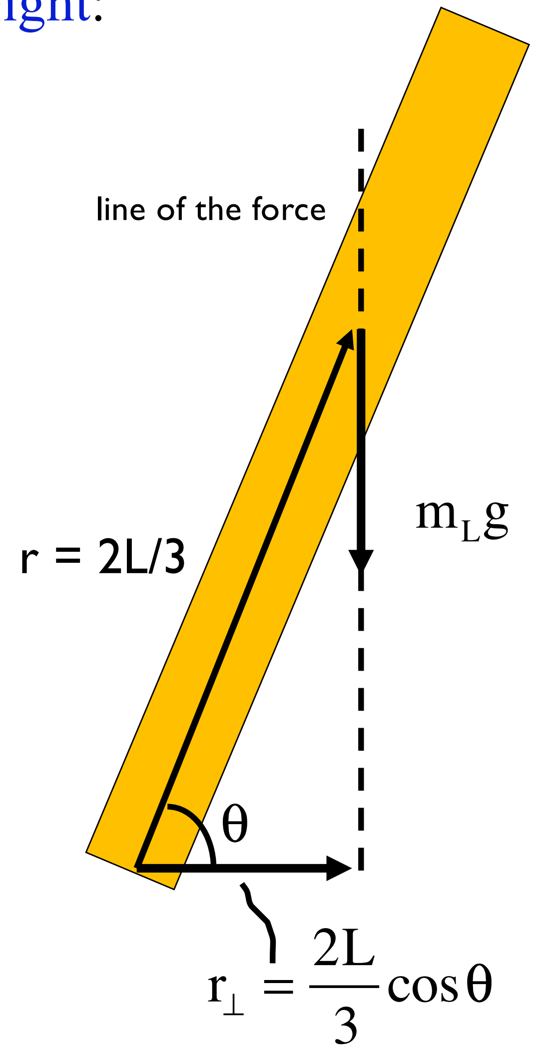
$$r_\perp = \frac{L}{2}\cos\theta$$

Note: The sign is negative as the force due to gravity on the ladder tends to make the ladder want to rotate clockwise, relative to the point of contact on the floor.

Note: $r_\perp$ is the shortest distance between the "line of force" and point about which torque is being taken . . .

To get the torque due to the man's weight:

$$\Gamma_{m_m g} = -\ |F|\ \ \ \ |r_\perp|$$
$$= -(m_m g)\left(\frac{2L}{3}\cos\theta\right)$$

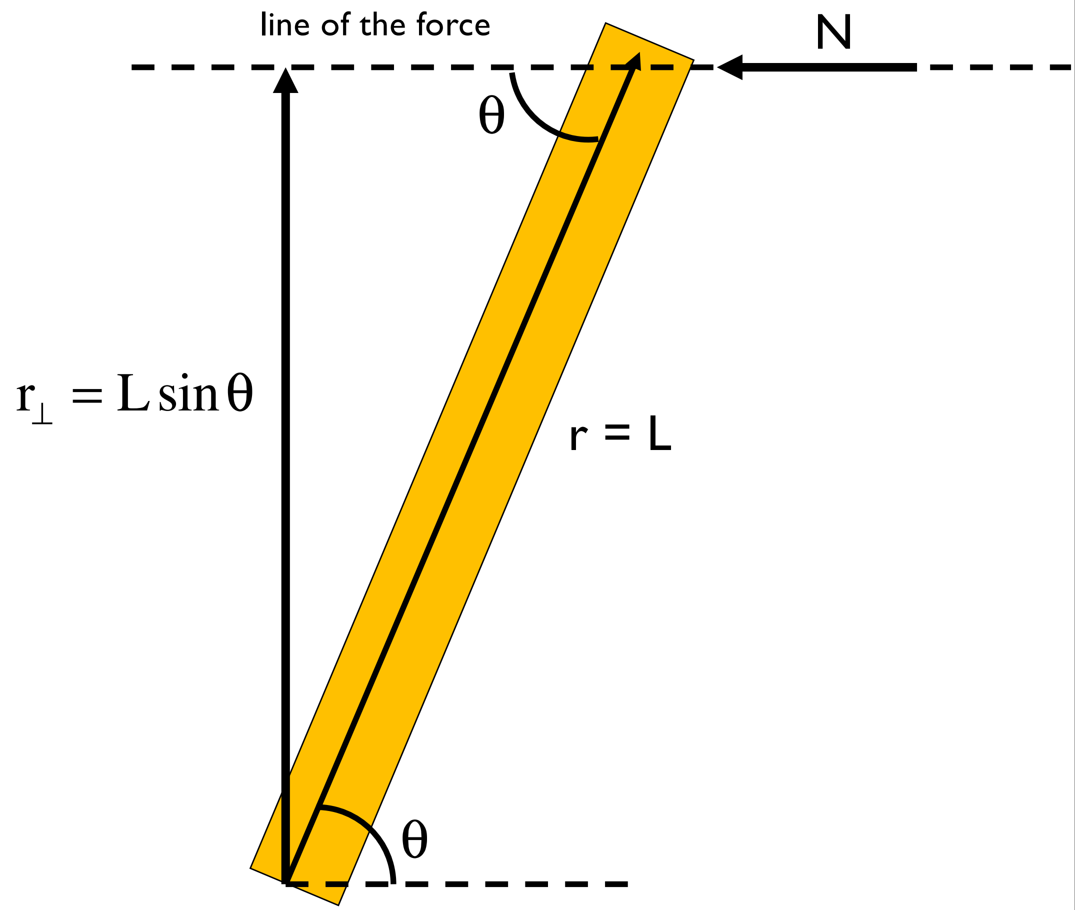


Note: The sign is negative as the force due to gravity on the man tends to make the ladder want to rotate clockwise, relative to the point of contact on the floor.

Note: $\mathbf{r}_\perp$ is the *shortest distance* between the "line of force" and point about which torque is being taken . . .

To get the torque due to "N:"

$$\Gamma_N = |F| \; |r_\perp| = (N)(L\sin\theta)$$



Note: The sign is positive as "N" makes the ladder want to rotate counterclockwise, relative to the point of contact on the floor.

Note: $r_\perp$ is the *shortest distance* between the "line of force" and point about which torque is being taken . . .

You wouldn't do all of those torque calculations separately on a test. You'd just write out what's found below. Doing that yields:

$\underline{\sum \Gamma_{cm}}$:

$$-\left|\Gamma_{m_L g}\right| \quad - \quad \left|\Gamma_{m_m g}\right| \quad + \quad \left|\Gamma_N\right| \quad + \left|\Gamma_H\right| + \left|\Gamma_V\right| = I\cancel{\alpha}^{=0}$$

$$-(m_L g)\left(\frac{\cancel{L}}{2}\right)\cos\theta - (m_m g)\left(\frac{2\cancel{L}}{3}\right)\cos\theta + N(\cancel{L}\sin\theta) + \ 0 \ + \ 0 \ = \ 0$$

$$\Rightarrow \quad N = \frac{\left(\frac{m_L g}{2}\right)\cos\theta + \left(\frac{2m_m g}{3}\right)\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad N = \left(\frac{2m_m g}{3} + \frac{m_L g}{2}\right)\cot\theta$$

Bottom line: This expression for "N" is exactly what we got when we summed up the forces in both the "x" and "y" directions, and then summed up the torques about the "center of mass" . . . except that has fewer steps and is easier . . .

Moral of the story:

When calculating and summing torques, pick your axis wisely.

– Remember that forces acting through the axis of rotation do not produce torques!

Remember the three ways to determine torque:

– $rF\sin\theta$, where you have to carefully find the angle between r and F
– rF-perpendicular, finding the component of F perpendicular to r
– r-perpendicular F (moment arm) – finding the r component

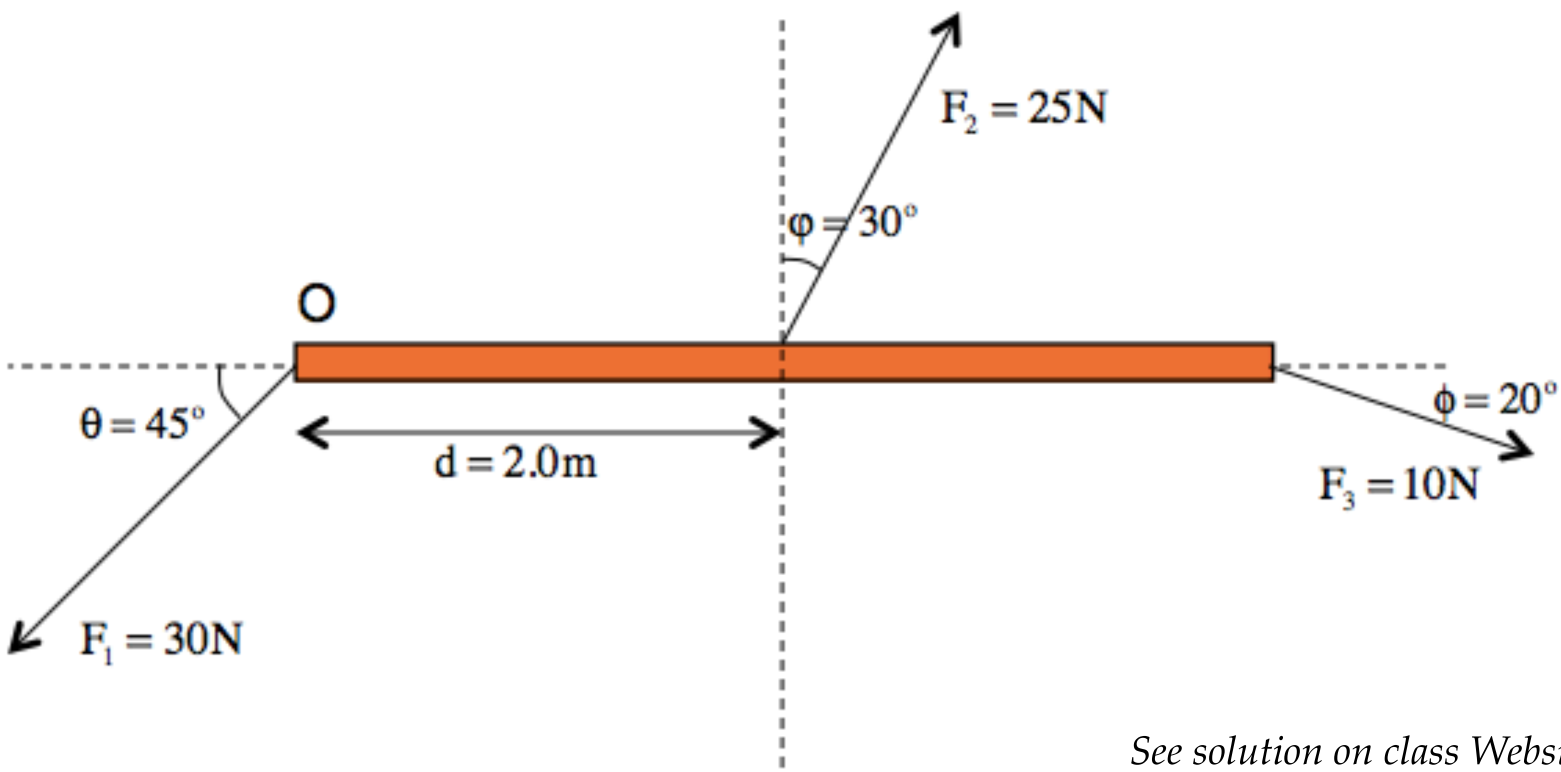
Which way you use just depends on the situation!

Another practice question (8.3)

For the system:

a.) Derive an expression for the net torque about "O" about an axis perpendicular to the page.

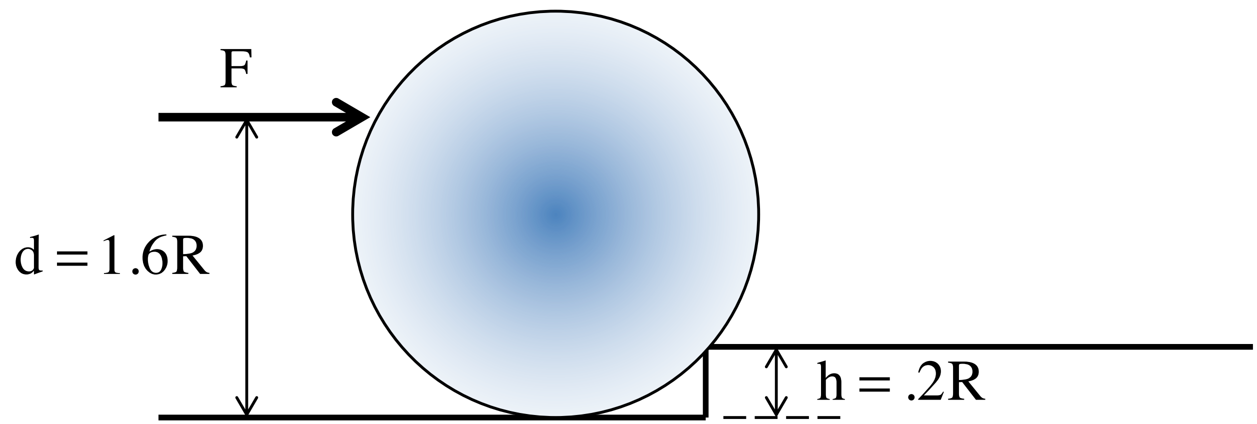
b.) Derive an expression for the net torque about "C" (the center of mass) about an axis perpendicular to the page.



See solution on class Website

One more example: wheel over a curb

A wheel of mass M and radius R is being pushed by a force F as shown over a small curb. The ball will rotate at the contact point with the curb to roll up and over.

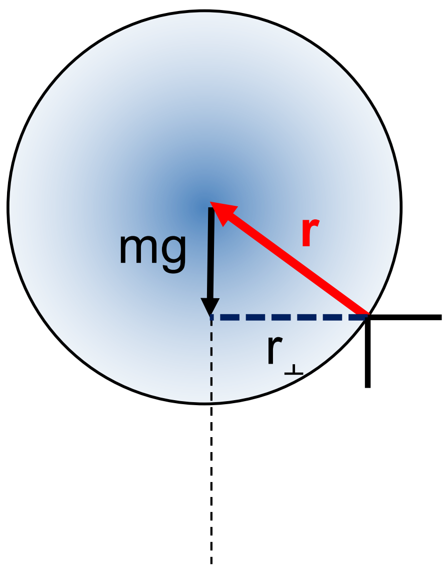


a) *Draw a free body* diagram for the ball at the instant it begins to roll up and over.

b) *Define and draw* "r" for the torque produced by gravity about the contact point. How about for the torque produced by F.

c) *Draw and determine* $r_\perp$ for the torque produced by gravity about the contact point. How about for the torque produced by F.

Wheel Over a Curb



To find r:

-*Draw a vector* the point about which you are taking your torque to the location of the force (from curb contact point to the center of mass, here)

To find $r_\perp$:

-extend the line of the force (mg)

-find the shortest distance between the axis of rotation (a point) and the line of the force. That's your $r_\perp$!

-to find its value here, use Pythagoras: hypotenuse is "R," left side is "R-h" so find the third side

This is where Quiz 2 stuff stops

What to know:

--How to define "r" vector;
--How to find angle between "r" vector and "F" vector;
--How to determine magnitude of a torque calculation;
--How to determine direction of a torque calculation;

Everything after this will not be on Monday's quiz, but will be on the quiz after . . .